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Complex positive maps and quaternionic unitary evolution

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Abstract
The complex projection of any n-dimensional quaternionic unitary dynamics defines a one-parameter positive semigroup dynamics. We show that the converse is also true, i.e. that any one-parameter positive semigroup dynamics of complex density matrices with maximal rank can be obtained as the complex projection of suitable quaternionic unitary dynamics.

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1. Introduction

It is well known that both classical and quantum phenomena can be described in terms of a set of propositions which form a lattice [1]. The lattice of a quantum theory can be embedded in a Hilbert space and the propositions are in correspondence with its subspaces [2]. The usual complex Hilbert space of quantum theory provides a realization for this structure, but more general realizations are admissible. In addition to the realization in terms of real Hilbert spaces [3], it has also been explored the structure of quantum mechanics on quaternionic Hilbert spaces [4, 5]. A systematic formulation of quaternionic quantum mechanics (QQM) can be given in [6].

In the past years, there have been some observations suggesting that quaternionic quantum mechanics may be useful to classify positive maps in complex quantum mechanics (CQM) [7]. Although the relation between completely positive maps and composite states is very well understood [8], the physical interpretation of the maps which are positive but not completely positive is still under investigation [9]; some hints in this direction can be derived from recent results by Kossakowski on quaternionic maps [7].

In CQM the subdynamics of a system in contact with an external environment is not, in general, unitary and in many cases can be described by a one-parameter positive semigroup of linear maps $\gamma_t = \exp(tL), t \geqslant 0$, with a generator L whose action on a density matrix mixed state ρ_α is given by [10–12]

$$L[\rho_\alpha] = -[H_\alpha, \rho_\alpha(t)] + \sum_{r,s=1}^{n^2-1} C_{rs} \left(F_r \rho_\alpha(t) F_s^\dagger - \frac{1}{2}\left\{F_r^\dagger F_s, \rho_\alpha(t)\right\}\right), \qquad (1)$$

in terms of its components: an anti-Hermitian Hamiltonian operator H_α, a family of $n^2 - 1$ traceless square matrices F_r, which form with the normalized identity $F_0 = I_n/\sqrt{n}$ an orthonormal set, i.e., $\mathrm{Tr}\left(F_r^\dagger F_s\right) = \delta_{rs}$, and a Hermitian matrix $[C_{rs}]$.[3]

L obeys the Lindblad–Kossakowski master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_\alpha = L[\rho_\alpha].$$

In this paper, we point out that the complex projection of any unitary dynamics on (right) quaternionic Hilbert spaces is controlled by the Lindblad–Kossakowski equation. Moreover, we also show that the inverse problem also holds under certain conditions. In particular, we prove that the complex dynamics controlled by equation (1) is the complex projection of a suitable quaternionic unitary dynamics for the dense set of maximal rank mixed states. For simplicity, we shall only consider finite-dimensional quantum systems.

This paper is organized as follows: in section 2 we discuss the density matrix formalism on quaternionic Hilbert spaces. Quaternionic positive maps are introduced in section 3. In section 4 we analyse the complex projection of the differential evolution equation of quaternionic unitary dynamics, while the inverse problem is considered in section 5. Finally, in section 6 we introduce some applications to two-level quantum systems and some concluding remarks are drawn in the last section.

2. Density matrix in QQM

Let us recall the basic elements of QQM. A (real) quaternion is usually expressed as

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where $q_l \in \mathbb{R}$ $(l = 0, 1, 2, 3)$, $i^2 = j^2 = k^2 = -1, ij = -ji = k$.

The quaternion skew-field $\mathbb{Q}$ is an algebra of rank 4 over $\mathbb{R}$, non-commutative and endowed with an involutory anti-automorphism (*conjugation*) such that

$$q \to \bar{q} = q_0 - q_1 i - q_2 j - q_3 k.$$

The real part of a quaternion q is defined by

$$\mathrm{Re}\, q = \tfrac{1}{2}(q + \bar{q}) = q_0.$$

In an (right) n-dimensional vector space over $\mathbb{Q}$, every linear operator is associated in a standard way with an $n \times n$ matrix acting on the left. Moreover, in analogy with the case of vector spaces over $\mathbb{C}$, one can introduce the notions of (right) quaternionic Hilbert space, unitarity and Hermiticity. The mathematical method to solve the (right) quaternionic eigenvalue problem can be found, for instance, in [13].

Let us denote by $M(\mathbb{Q})$ and $M(\mathbb{C})$ the space of $n \times m$ quaternionic and complex matrices, respectively. The complex projection M_α of any matrix $M \in M(\mathbb{Q})$ can be defined by means of the projection operator

$$P : M(\mathbb{Q}) \to M(\mathbb{C})$$

[3] γ_t is completely positive if and only if this matrix $[C_{rs}]$ is positive and $\mathrm{Tr}\, H_\alpha = 0$ [11].

given by

$$P(M) = \tfrac{1}{2}(M - \mathrm{i}Mi) = M_\alpha.$$

In a similar manner the quaternionic part $M_\beta \in M(\mathbb{C})$ of M can be defined from the relation $M - P(M) = \mathrm{j}M_\beta$. It is, then, obvious that any matrix $M \in M(\mathbb{Q})$ can be univocally split into its complex and quaternionic components $M = M_\alpha + \mathrm{j}M_\beta \in M(\mathbb{Q})$.

The density matrix ρ_ψ associated with a pure state $|\psi\rangle = |\psi_\alpha\rangle + \mathrm{j}|\psi_\beta\rangle$ belonging to a right quaternionic Hilbert space $\mathcal{H}^{\mathbb{Q}}$ is defined by

$$\rho_\psi = |\psi\rangle\langle\psi| = |\psi_\alpha\rangle\langle\psi_\alpha| + (|\psi_\beta\rangle\langle\psi_\beta|)^* + \mathrm{j}[|\psi_\beta\rangle\langle\psi_\alpha| - (|\psi_\alpha\rangle\langle\psi_\beta|)^*] \tag{2}$$

(where $*$ denotes complex conjugation and $|\psi_\alpha\rangle, |\psi_\beta\rangle$ are complex vectors) and is the same for all normalized ray representatives. By definition, density matrices ρ_ψ associated with pure states are represented by rank-1 positive definite quaternionic Hermitian operators of $\mathcal{H}^{\mathbb{Q}}$ with unit trace. In analogy with complex mixed states one can introduce the notion of quaternionic mixed states, which are defined by positive definite quaternionic Hermitian operators ρ of $\mathcal{H}^{\mathbb{Q}}$ with unit trace and rank grater than one. Positivity implies that for any pure state $|\psi\rangle \in \mathcal{H}^{\mathbb{Q}}$

$$\langle\rho\rangle_\psi = \langle\psi|\rho|\psi\rangle = \mathrm{Re}\,\mathrm{Tr}(\rho|\psi\rangle\langle\psi|) = \mathrm{Re}\,\mathrm{Tr}(\rho\rho_\psi) > 0, \tag{3}$$

where $\mathrm{Re}\,\mathrm{Tr}\,C = \mathrm{Re}\big(\sum_r C_{rr}\big) = \sum_r \mathrm{Re}\,C_{rr}$ denotes the real part of the trace of the operator C (this real trace enjoys the cyclic property $\mathrm{Re}\,\mathrm{Tr}\,AB = \mathrm{Re}\,\mathrm{Tr}\,BA$ [4]).

The expectation value of a quaternionic Hermitian operator A on a state $|\psi\rangle$ can be expressed in terms of ρ_ψ as [6]

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle = \mathrm{Re}\,\mathrm{Tr}(A|\psi\rangle\langle\psi|) = \mathrm{Re}\,\mathrm{Tr}(A\rho_\psi). \tag{4}$$

In a similar manner one can define the expectation value of a quaternionic Hermitian operator A on a mixed quaternionic state ρ by

$$\langle A\rangle_\rho = \mathrm{Re}\,\mathrm{Tr}(A\rho). \tag{5}$$

Now, expanding $A = A_\alpha + \mathrm{j}A_\beta$ and $\rho = \rho_\alpha + \mathrm{j}\rho_\beta$ in terms of complex matrices $A_\alpha, A_\beta, \rho_\alpha$ and ρ_β, it follows that the expectation value $\langle A\rangle_\psi$ may depend on A_β or ρ_β only if both A_β and ρ_β are different from zero. Indeed,

$$\langle A\rangle_\rho = \mathrm{Re}\,\mathrm{Tr}(A\rho) = \mathrm{Re}\,\mathrm{Tr}(A_\alpha\rho_\alpha - A_\beta^*\rho_\beta). \tag{6}$$

Thus, the expectation value of an Hermitian operator A on the state ρ depends on the quaternionic parts of A and ρ, only if both the observable and the state are represented by genuine quaternionic matrices.

However, if an observable O is described by a pure *complex* Hermitian matrix, its expectation value does not depend on the quaternionic part $\mathrm{j}\rho_\beta$ of the state $\rho = \rho_\alpha + \mathrm{j}\rho_\beta$. Moreover, *the expectation value predicted in the standard (complex) quantum mechanics for the state ρ_α coincides with that predicted in quaternionic quantum mechanics for the state ρ*, since

$$\mathrm{Tr}(O\rho_\alpha) = \mathrm{Re}\,\mathrm{Tr}(O\rho_\alpha) = \mathrm{Re}\,\mathrm{Tr}(O\rho).$$

The time evolution equation for ρ_ψ reads

$$\frac{\partial}{\partial t}\rho_\psi = -[H, \rho_\psi], \tag{7}$$

($\hbar = 1$) where H is the quaternionic anti-Hermitian Hamiltonian operator [6].

The time evolution equation of $\langle A\rangle_\psi$ is given by

$$\frac{\partial}{\partial t}\langle A\rangle_\psi = \mathrm{Re}\,\mathrm{Tr}\left\{\left(\frac{\partial A}{\partial t} + [H, A]\right)\rho_\psi\right\} = \left\langle\frac{\partial A}{\partial t} + [H, A]\right\rangle_\psi.$$

3. Complex and quaternionic maps

In CQM as well as in QQM, the *unitary* evolution of a pure state ρ is described by a *completely positive* map:

$$\rho \to \rho' = U\rho U^\dagger$$

where U is unitary. Thus, $\rho^2 = \rho$ implies $\rho'^2 = \rho'$.

However, if one considers in CQM the reduced matrices associated with two subsystems of an entangled state, the evolution is described by a non-unitary positive map, which preserves the Hermiticity of the subsystem mixed states. In particular, in the two-dimensional case, any such a positive map is *decomposable*, i.e., it is the sum of completely positive and completely copositive maps and in the simplest case we get

$$\rho \to \rho' = U_\alpha \rho U_\alpha^\dagger + U_\beta \rho^T U_\beta^\dagger \tag{8}$$

where U_α, U_β need not to be unitary and T denotes the transposition operation.

A remarkable result, due to Kossakowski [7], establishes that any complex decomposable map of a complex density matrix can be considered as the complex projection of a corresponding quaternionic completely positive map of the same complex density matrix. This result follows from the observation that for any unitary quaternionic operator $U = U_\alpha + U_\beta j$ and any complex density matrix ρ_α we have that

$$U\rho_\alpha U^\dagger = U_\alpha \rho_\alpha U_\alpha^\dagger + U_\beta \rho_\alpha^T U_\beta^\dagger + \mathrm{j}\big(U_\beta^* \rho_\alpha U_\alpha^\dagger - U_\alpha^* \rho_\alpha^T U_\beta^\dagger\big). \tag{9}$$

Let us illustrate this result in a simple two-qubit system. The evolution of the composite system $\mathcal{H}^{\mathbb{C}} = \mathbb{C}^2 \otimes \mathbb{C}^2$ is given by a unitary transformation

$$|+,-\rangle \to c_1|+,-\rangle + c_2|-,+\rangle, \qquad c_1, c_2 \in \mathbb{C}. \tag{10}$$

The density operators $\rho_\alpha^{(1)}$ and $\rho_\alpha^{(2)}$ of the subsystems can be obtained by taking the partial trace of the density matrices associated with the states in equation (10) with respect to the subsystems 2 and 1 respectively,

$$\rho_\alpha^{(1)}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_\alpha^{(1)}(t) = \begin{pmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{pmatrix}$$

and

$$\rho_\alpha^{(2)}(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_\alpha^{(2)}(t) = \begin{pmatrix} |c_2|^2 & 0 \\ 0 & |c_1|^2 \end{pmatrix}.$$

Thus, the dynamics inherited by each subsystem transforms a pure into a genuine mixed state. The dynamical evolution of the subsystems is quite different from the dynamics described by a unitary evolution operator on their Hilbert spaces.

In order to discuss the same physical system in a quaternionic Hilbert space, we first observe that in $\mathcal{H}^{\mathbb{C}}$ as well as in $\mathcal{H}^{\mathbb{Q}}$ the spin observables are represented by the complex Hermitian matrices [6]

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, in $\mathcal{H}^{\mathbb{Q}}$ the expectation values of the spin observables do not depend on the genuine quaternionic part $\mathrm{j}\rho_\beta$ of the state $\rho = \rho_\alpha + \mathrm{j}\rho_\beta$. Now, there exists a quaternionic unitary operator U

$$U = \begin{pmatrix} |c_1| & |c_2| \\ |c_2|\mathrm{j} & -|c_1|\mathrm{j} \end{pmatrix}, \qquad U^\dagger = \begin{pmatrix} |c_1| & -|c_2|\mathrm{j} \\ |c_2| & |c_1|\mathrm{j} \end{pmatrix}, \tag{11}$$

such that

$$\rho^{(1)}(t) = U\rho^{(1)}(0)U^{\dagger} = \rho_{\alpha}^{(1)}(t) + \begin{pmatrix} 0 & -|c_1||c_2|\mathrm{j} \\ |c_1||c_2|\mathrm{j} & 0 \end{pmatrix} \tag{12}$$

and

$$\rho^{(2)}(t) = U\rho^{(2)}(0)U^{\dagger} = \rho_{\alpha}^{(2)}(t) + \begin{pmatrix} 0 & |c_1||c_2|\mathrm{j} \\ -|c_1||c_2|\mathrm{j} & 0 \end{pmatrix}, \tag{13}$$

where $\rho_{\alpha}^{(1)}(t)$ and $\rho_{\alpha}^{(2)}(t)$ represent the complex projections of the quaternionic pure states $\rho^{(1)}(t)$ and $\rho^{(2)}(t)$, respectively.

This simple example raises two natural questions:

(1) Are the complex projections of quaternionic unitary dynamics controlled by the Lindblad–Kossakowski equation?

(2) Is it possible to obtain the complex dynamics controlled by the Lindblad–Kossakowski equation as the complex projection of suitable quaternionic unitary dynamics?

The answers to these questions are provided in sections 4 and 5, respectively.

4. Complex projection evolution

In this section we shall consider quaternionic unitary dynamics and we will show that the corresponding complex projection is indeed controlled by equation (1).

Let

$$\rho(t) = U(t)\rho(0)U^{\dagger}(t) \tag{14}$$

be the unitary evolution equation of a quaternionic density matrix ρ, where

$$U(t) = (U_{\alpha} + U_{\beta}j)(t) = T_o \exp\left(-\int_0^t \mathrm{d}u\, H(u)\right) \tag{15}$$

and T_o denotes as usual the time ordering operator.

The complex projection of a quaternionic density matrix satisfies the following proposition:

Proposition 1. *The complex projection ρ_{α} of any quaternionic positive (semi)definite Hermitian matrix $\rho = \rho_{\alpha} + \mathrm{j}\rho_{\beta}$ is positive (semi)definite Hermitian.*

Proof. If ρ is a quaternionic positive (semi)definite Hermitian matrix, then there exists a suitable matrix $\eta = \eta_{\alpha} + \mathrm{j}\eta_{\beta}$ such that $\rho = \eta\eta^{\dagger} = \eta_{\alpha}\eta_{\alpha}^{\dagger} + \left(\eta_{\beta}\eta_{\beta}^{\dagger}\right)^{*} + \mathrm{j}\left(\eta_{\beta}\eta_{\alpha}^{\dagger} - \eta_{\alpha}^{*}\eta_{\beta}^{T}\right)$ [14]. Moreover, a well-known property of complex positive (semi)definite matrices ensures that any non-negative linear combination of them is positive (semi)definite [15]. Therefore, $\rho_{\alpha} = \eta_{\alpha}\eta_{\alpha}^{\dagger} + \left(\eta_{\beta}\eta_{\beta}^{\dagger}\right)^{*}$ is a positive (semi)definite Hermitian. □

Moreover, the quaternionic density matrix $U\rho U^{\dagger}$ can be expanded as

$$\begin{aligned} U\rho U^{\dagger} = {} & U_{\alpha}\rho_{\alpha}U_{\alpha}^{\dagger} + U_{\beta}\rho_{\alpha}^{T}U_{\beta}^{\dagger} - U_{\beta}\rho_{\beta}U_{\alpha}^{\dagger} + U_{\alpha}\rho_{\beta}^{*}U_{\beta}^{\dagger} \\ & + \mathrm{j}\left(U_{\beta}^{*}\rho_{\alpha}U_{\alpha}^{\dagger} - U_{\alpha}^{*}\rho_{\alpha}^{T}U_{\beta}^{\dagger} + U_{\alpha}^{*}\rho_{\beta}U_{\alpha}^{\dagger} + U_{\beta}^{*}\rho_{\beta}^{*}U_{\beta}^{\dagger}\right) \end{aligned} \tag{16}$$

in terms of its complex and purely quaternionic components.

Now, as a direct consequence of proposition 1, it follows that the complex map

$$\rho_{\alpha} \to U_{\alpha}\rho_{\alpha}U_{\alpha}^{\dagger} + U_{\beta}\rho_{\alpha}^{T}U_{\beta}^{\dagger} - U_{\beta}\rho_{\beta}U_{\alpha}^{\dagger} + U_{\alpha}\rho_{\beta}^{*}U_{\beta}^{\dagger} \tag{17}$$

is a positive map and the composition of any two such maps is also a positive map.

Thus, we have shown that *to every quaternionic unitary evolution corresponds an evolution of its complex projection generated by a corresponding semigroup of complex positive maps*.

Let us consider now the differential evolution equation for the complex projection density matrix $\rho_\alpha(t)$ associated with a quaternionic density matrix $\rho(t)$. When we consider time-dependent quaternionic unitary dynamics, the differential equation associated with the time evolution for ρ reads

$$\frac{\partial}{\partial t}\rho(t) = -[H(t), \rho(t)], \tag{18}$$

where

$$H(t) = -\left(\frac{\partial}{\partial t}U(t)\right)U^\dagger(t) \tag{19}$$

is the quaternionic time-dependent anti-Hermitian Hamiltonian operator (see equation (15)).

If we introduce the splittings $\rho = \rho_\alpha + \mathrm{j}\rho_\beta$ and $H = H_\alpha + \mathrm{j}H_\beta$ into equation (18), and extract the complex part from the purely quaternionic ones, the evolution equation for the complex projection density matrix reduces to

$$\frac{\partial}{\partial t}\rho_\alpha = -[H_\alpha, \rho_\alpha] + H_\beta^*\rho_\beta - \rho_\beta^* H_\beta \tag{20}$$

where $H_\alpha^\dagger = -H_\alpha$, $H_\beta^T = H_\beta$ and $\rho_\beta^T = -\rho_\beta$.

Note that from the linearity of equation (18) in the space of quaternionic density matrices ρ it follows immediately the linearity of equation (20) in the space of complex density matrices ρ_α.

It is straightforward to see that the term $H_\beta^*\rho_\beta - \rho_\beta^* H_\beta$ in equation (20) is Hermitian and traceless (as consequence of the cyclic property of the trace), like the non-Hamiltonian term,

$$L_i[\rho_\alpha] = \sum_{r,s=1}^{n^2-1} C_{rs}\left(F_r\rho_\alpha(t)F_s^\dagger - \frac{1}{2}\left\{F_r^\dagger F_s, \rho_\alpha(t)\right\}\right), \tag{21}$$

in the generator of one-parameter positive dynamical semigroups given in equation (1). Therefore, we have shown that *the complex projection of any quaternionic (time-dependent) unitary dynamics belongs to the set of complex dynamics described by the Lindblad–Kossakowski equation*.

Remark 1. As a direct consequence of equation (16) it follows that the complex positive map in equation (17) is decomposable if $U_\beta\rho_\beta U_\alpha^\dagger = U_\alpha\rho_\beta^* U_\beta^\dagger$. The converse is, however, not true. In fact, for the complex projection of the map induced by the quaternionic unitary matrix (11) on the density matrix in equation (12), it turns out that $U_\beta\rho_\beta U_\alpha^\dagger \neq U_\alpha\rho_\beta^* U_\beta^\dagger$.

Remark 2. The dynamical problem equation (19) associated with a given Hamiltonian $H(t)$ can be solved by standard techniques if it is reformulated in a complex framework [16]. In fact, by using the isomorphism between quaternionic matrices and their corresponding complex matrices which also preserves unitarity [14], equation (19) can be rewritten in a double-dimensional complex space as follows:

$$\begin{pmatrix} H_\alpha(t) & H_\beta(t) \\ -H_\beta^*(t) & H_\alpha^*(t) \end{pmatrix} = -\frac{\partial}{\partial t}\begin{pmatrix} U_\alpha(t) & U_\beta(t) \\ -U_\beta^*(t) & U_\alpha^*(t) \end{pmatrix}\begin{pmatrix} U_\alpha(t) & U_\beta(t) \\ -U_\beta^*(t) & U_\alpha^*(t) \end{pmatrix}^\dagger.$$

5. Inverse problem

A natural question arises: is there a solution of the converse problem? Can any complex dynamics governed by the Lindblad–Kossakowski equation be considered as the projection of a quaternionic unitary dynamics? The answer to these questions is the goal of this section.

A necessary condition is that the traceless Hermitian term (21) be decomposable into symmetric and skew-symmetric parts as the traceless Hermitian term in equation (20). The following proposition shows that it is also a sufficient condition.

Proposition 2. *For any complex Hermitian traceless matrix K of dimension n there exist a real skew-symmetric matrix R and a complex symmetric matrix S such that K can be decomposed as*

$$K = SR - RS^{*}. \tag{22}$$

Proof. If the condition (22) holds

$$\operatorname{Tr} K = \operatorname{Tr}(SR - RS^{*}) = \operatorname{Tr}(SR) - \operatorname{Tr}(S^{*}R) = \operatorname{Tr}(SR) - \operatorname{Tr}(SR)^{*}$$

which would imply that $\operatorname{Tr} K$ is purely imaginary. But since K is Hermitian $\operatorname{Tr} K$ should be real. This means that $\operatorname{Tr} K = 0$.

Let K be any traceless Hamiltonian operator. K can be decomposed into its real and pure imaginary terms $K = K_0 + \mathrm{i}K_1$. The decomposition is preserved by real orthogonal similarity transformations Q. Thus, the real component K_0 can be always reduced to its diagonal form

$$\Lambda = QK_0Q^{T} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, -\lambda_1 - \lambda_2 - \cdots - \lambda_{n-1}),$$

by means of such transformations, and $QKQ^{T} = \Lambda + \mathrm{i}\Omega$, with $\Omega = QK_1Q^{T}$. The splitting (22) can then be rewritten in the following form

$$\Lambda + \mathrm{i}\Omega = [\Sigma_0, \Upsilon] + \mathrm{i}\{\Sigma_1, \Upsilon\}, \tag{23}$$

where $\Upsilon = QRQ^{T}$ and Σ_0 and Σ_1 denote the real and the imaginary parts of $QSQ^{T} = \Sigma_0 + \mathrm{i}\Sigma_1$.

Let us now consider the solution of the matrix equation

$$\Lambda = [\Sigma_0, \Upsilon], \tag{24}$$

with unknown matrices Σ_0 and Υ.

A solution of equation (24) always exists. In fact, a simple solution is provided by the skew-symmetric matrix

$$\Upsilon = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{pmatrix},$$

and the real symmetric matrix $\Sigma_0 = (\Sigma_0)_{lm}$ whose entries are

$$(\Sigma_0)_{ll} = \frac{1}{2}\sum_{p=1}^{l-1}\lambda_p - \frac{1}{2}\sum_{p=l+1}^{n-1}\lambda_p,$$

$$(\Sigma_0)_{lm} = -(\Sigma_0)_{ml} = \frac{\delta_{lm-1}}{2}\sum_{p=m}^{n-1}\lambda_p \qquad n > m > l$$

$$(\Sigma_0)_{1n} = -(\Sigma_0)_{n1} = -\frac{1}{2}\sum_{p=1}^{n-1}\lambda_p.$$

For real K matrices the proposition is, thus, already proven. Let us analyse the solution for any Hermitian traceless matrix K. In such a case we have to find also a solution of the equation

$$\Omega = \{\Sigma_1, \Upsilon\}. \tag{25}$$

First, we note that the spectrum of Υ is non-degenerate. Indeed, for any eigenvalue λ of Υ the characteristic matrix $\Upsilon - \lambda\mathbb{I}$ has rank $n-1$. This follows from the fact that the $n-1$ vectors

$$\begin{array}{lllllll} e_2 = (-1, & -\lambda, & 1, & \cdots & 1, & 1) \\ e_3 = (-1, & -1, & -\lambda, & \cdots & 1, & 1) \\ e_4 = (-1, & -1, & -1, & \cdots & 1, & 1) \\ \vdots \quad \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{n-1} = (-1, & -1, & -1, & \cdots & -\lambda, & 1) \\ e_n = (-1, & -1, & -1, & \cdots & -1, & -\lambda) \end{array} \tag{26}$$

are linear independent for any value of λ. Indeed, for any linear combination

$$\sum_{i=2}^{n} a_i e_i = 0,$$

we have that

$$\sum_{i=2}^{n} a_i = 0, \qquad \sum_{i=2}^{n-1} a_i = \lambda a_n \qquad \text{and} \qquad \sum_{i=3}^{n} a_i = -\lambda a_2.$$

Thus, $a_n = -\lambda a_n$ and $a_2 = \lambda a_2$ which implies that either a_2 or a_n vanish and $\lambda = 1$ or $\lambda = -1$. Iterating the argument one concludes that $a_i = 0$ for any $i = 2, 3, \ldots, n$. Consequently all eingenvalues λ_i of Υ are different $\lambda_i \neq \lambda_j$ if $i \neq j$. These eigenvalues are either 0 or pure imaginary because of the antisymmetry property of Υ. In the even case all eigenvalues are pure imaginary and they form dual pairs $\lambda_\pm = \pm \mathrm{i}\mu$. In the odd-dimensional case there is an extra zero mode. In the first case the matrix Υ defines a symplectic form and in the second one a contact form.

By an orthogonal similarity transformation (which does not modify the decomposition of Ω and hence of K into skew-symmetric and symmetric parts) we reduce Υ to its canonical form in Darboux coordinates

$$\Upsilon = \begin{pmatrix} \mathrm{i}\mu_1\sigma_2 & 0 & \cdots & 0 \\ 0 & \mathrm{i}\mu_2\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{i}\mu_f\sigma_2 \end{pmatrix} \tag{27}$$

for even Υ dimension $n = 2f$ and

$$\Upsilon = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathrm{i}\mu_1\sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \mathrm{i}\mu_2\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{i}\mu_f\sigma_2 \end{pmatrix} \tag{28}$$

for odd Υ dimension $n = 2f + 1$, where

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{29}$$

Let us consider from now on n odd. For the even case the proof is obtained by the same method killing the first row and column of Υ in equation (28).

Now, equation (25) can be considered as a mapping from the space of symmetric real matrices Σ_1 into that of skew-symmetric real matrices Ω. The kernel of such a map is provided by the matrices Σ_1 which commute with Υ. The most general form of such matrices is

$$\Sigma_1 = \begin{pmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{12} & m^{\alpha}_{22}\sigma_\alpha & m^{\alpha}_{23}\sigma_\alpha & \cdots & m^{\alpha}_{2f}\sigma_\alpha \\ m_{13} & m^{\alpha}_{32}\sigma_\alpha & m^{\alpha}_{33}\sigma_\alpha & \cdots & m^{\alpha}_{3f}\sigma_\alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{1n} & m^{\alpha}_{f2}\sigma_\alpha & m^{\alpha}_{f3}\sigma_\alpha & \cdots & m^{\alpha}_{ff}\sigma_\alpha \end{pmatrix} \tag{30}$$

where $\alpha = 0, 1, 2, 3,$

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and m^{α}_{ij} are real parameters satisfying the constraints $m^2_{ii} = 0, m^2_{ij} = -m^2_{ji}$ and $m^{\alpha}_{ij} = m^{\alpha}_{ji}$ for $\alpha \neq 2$.

Now

$$\{\Sigma_1, \Upsilon\} = 0$$

if and only if $m_{ij} = 0$ and $m^{\alpha}_{ij} = 0$ for $i \neq j$, and $m^0_{ii} = 0$. In particular this means that the kernel of the map (25) is defined by the matrices of the form

$$\Sigma_1 = \begin{pmatrix} m_{11} & 0 & 0 & \cdots & 0 \\ 0 & m^1_{22}\sigma_1 + m^3_{22}\sigma_3 & 0 & \cdots & 0 \\ 0 & 0 & m^1_{33}\sigma_1 + m^3_{33}\sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m^1_{ff}\sigma_1 + m^3_{ff}\sigma_3 \end{pmatrix}.$$

The dimension of this kernel is n. The remaining eigenvalues of the map defined by equation (25) are $\lambda_i - \lambda_j$ for $i \neq j$ and do not vanish because of the non-degenerate character of the matrix Υ.

The dimension of the space of (real) symmetric matrices Σ_1 is $(n+1/2)$ and that of skew-symmetric real matrices Ω is $(n/2)$. Since the kernel of the map is $n = (n+1/2) - (n/2)$ dimensional it follows that the map is surjective, i.e. for any skew-symmetric real matrix Ω there always exists a symmetric matrix Σ_1 satisfying equation (25), which completes the proof[4]. □

A consequence of the above proposition is that any Hermitian term (21) is decomposable into symmetric and skew-symmetric parts as the Hermitian term in equation (20).

[4] An exhaustive characterization of the solutions of matrix equations similar to (22) can be found in [17].

Moreover, there is an extra property of the non-Hamiltonian term of Lindblad–Kossakowski evolution (1) which this decomposition preserves. It is the linear convexity,

$$L\big(c\rho_{\alpha_1} + (1-c)\rho_{\alpha_0}\big) = cL\big(\rho_{\alpha_1}\big) + (1-c)L\big(\rho_{\alpha_0}\big). \tag{31}$$

Indeed, the complex projection of the quaternionic state

$$\rho_c = c\rho_{\alpha_1} + (1-c)\rho_{\alpha_0} + \mathrm{j}R \tag{32}$$

under the quaternionic unitary evolution generated by $H = H_\alpha + \mathrm{j}S_\alpha$ also verifies that $\rho_c = c\rho_1 + (1-c)\rho_0$, with $\rho_1 = \rho_{\alpha_1} + \mathrm{j}R$ and $\rho_0 = \rho_{\alpha_0} + \mathrm{j}R$; and $K_c = S_c R - RS_c^*$ with $S_c = cS_1 + (1-c)S_0$ and $L\big(c\rho_{\alpha_1} + (1-c)\rho_{\alpha_0}\big)$.

However, the decomposition (22) is not unique. In fact, for any matrix $G \in \{G\}$ of the complex non-singular symmetric commutant[5] of S and R, the symmetric and skew-symmetric matrices given respectively by $S' = SG^{-1}$ and $R' = GR$ also satisfy the equation

$$K = S'R' - R'^*S'^*. \tag{33}$$

This property can be used to solve the last obstacle in the formulation of the dissipative evolution governed by (1) as the complex projection of quaternionic unitary dynamics. By construction it is obvious that the evolution induced by $H(t) = H_\alpha(t) + \mathrm{j}S(t)$ on the quaternionic matrix $\rho = \rho_\alpha + \mathrm{j}R$ projects to a complex evolution of the matrix ρ_α governed by the Lindblad–Kossakowski generator (1). The problem is that even if ρ_α corresponds to a density matrix of a mixed state, ρ does not inherit the same property. In particular, the positive definiteness of ρ is not guaranteed by construction. However, one might use the freedom in the choice of R and S matrices to get new $\rho = \rho_\alpha + \mathrm{j}R'$ and $H(t) = H_\alpha(t) + \mathrm{j}S'(t)$ such that the matrix ρ fulfils all physical requirements.

The following lemma shows under which conditions the quaternionic dynamics projects into a consistent complex dynamics.

Lemma 1. *Let ρ_α be a complex positive (semi)definite Hermitian matrix. Then, there exists a complex skew-symmetric matrix ρ_β such that the quaternionic Hermitian matrix $\rho = \rho_\alpha + \mathrm{j}\rho_\beta$ is positive (semi)definite if and only if* rank $\rho_\alpha > 1$.

Proof. Let rank $\rho_\alpha > 1$. Without loss of generality we can assume ρ_α to be diagonal: $\rho_\alpha = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$. The choice

$$\rho_\beta = \begin{pmatrix} 0 & r & 0 & \cdots \\ -r & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with $\lambda_1\lambda_2 \geqslant r^2$, guarantees the positivity of $\rho = \rho_\alpha + \mathrm{j}\rho_\beta$ [13].

If rank $\rho_\alpha = 1$ we can consider the Hermitian complex component

$$\widetilde{\rho} = \widetilde{\rho}_\alpha + \widetilde{j\rho}_\beta = \begin{pmatrix} \rho_\alpha & \rho_\beta \\ -\rho_\beta^* & \rho_\alpha^* \end{pmatrix}$$

of ρ [14] with $\rho_\alpha = \mathrm{diag}(1, 0, \ldots, 0)$. The complex Hermitian matrix

$$\widetilde{j\rho}_\beta = \begin{pmatrix} 0 & \rho_\beta \\ -\rho_\beta^* & 0 \end{pmatrix}$$

[5] The commutant is defined by $\{G\} = \{G, G = G^T, \det G \neq 0, [G, S] = [G, R] = 0\}$. Note that $\{G\}$ is non-void, in fact any complex multiple of the identity matrix belongs to it.

presents (real) even degenerate eigenvalues [14] in pairs $\lambda, -\lambda$. Its characteristic equation reads [14]:

$$\det\begin{pmatrix} -\lambda & \rho_\beta \\ -\rho_\beta^* & -\lambda \end{pmatrix} = \det[\lambda^2 - \rho_\beta {\rho_\beta}^\dagger] = \det[\lambda^2 - D^2],$$

where D^2 denotes the diagonal form of the complex positive (semi)definite Hermitian matrix $\rho_\beta {\rho_\beta}^\dagger$.

The positive (semi)definite character of $\widetilde{\rho}$ implies $\lambda_l(\widetilde{\rho}_\alpha) \geqslant \lambda_l(\widetilde{j\rho_\beta})$ for all $l = 1, 2, \ldots, 2n$ [15], but we have shown that the (even degenerate) eigenvalues $\lambda_l(\widetilde{j\rho_\beta})$ appear in pairs $\lambda, -\lambda$, thus $\widetilde{j\rho_\beta} = 0$, and $\rho = \rho_\alpha$. □

A consequence of the previous lemma is that any (complex) Lindblad–Kossakowski dynamics of complex pure states cannot be derived as the complex projection of quaternionic unitary dynamics.

The most general result is provided by the following proposition.

Proposition 3. *Let ρ_α be a complex positive definite Hermitian matrix. For any skew-symmetric complex matrix ρ_β there exists a real parameter $\epsilon > 0$ such that the quaternionic Hermitian matrix $\rho = \rho_\alpha + \epsilon \mathrm{j}\rho_\beta$ is positive (semi)definite.*

Proof. By the correspondence between quaternionic matrices ρ and their complex counterparts $\widetilde{\rho}$ [14], it is enough to prove that the complex component

$$\widetilde{\rho} = \widetilde{\rho}_\alpha + \widetilde{\epsilon j \rho_\beta} = \begin{pmatrix} \rho_\alpha & \epsilon\rho_\beta \\ -\epsilon\rho_\beta^* & \rho_\alpha^* \end{pmatrix}$$

of ρ is positive (semi)definite for suitable values of the real parameter ϵ. As we have shown in the proof of the previous lemma, the complex Hermitian matrix

$$\widetilde{j\rho_\beta} = \begin{pmatrix} 0 & \rho_\beta \\ -\rho_\beta^* & 0 \end{pmatrix}$$

has (real even degenerate) eigenvalues in pairs $\lambda, -\lambda$. Now, as direct consequence of the Weyl theorem [15] and the positivity of $\widetilde{\rho}_\alpha$ we have $\epsilon\lambda_l(\widetilde{\mathrm{j}\rho_\beta}) + \lambda_1(\widetilde{\rho}_\alpha) \leqslant \lambda_l(\widetilde{\rho}_\alpha + \epsilon\widetilde{j\rho_\beta}) = \lambda_l(\widetilde{\rho}), l = 1, \ldots, 2n, \lambda_1(\widetilde{\rho}_\alpha) = \min\{\lambda_l(\widetilde{\rho}_\alpha)\} > 0$. Then, by choosing$\epsilon$ such that $\epsilon \max\{\lambda_l(\widetilde{j\rho_\beta})\} \leqslant \lambda_1(\widetilde{\rho}_\alpha)$ we immediately obtain that $\widetilde{\rho}$ is positive (semi)definite. □

In such a case, we have shown that the one-parameter positive semigroup dynamics with generator $L[\rho_\alpha] = -[H_\alpha, \rho_\alpha] + L_i[\rho_\alpha]$ of *positive definite* complex density matrices ρ_α is the complex projection of quaternionic unitary dynamics of quaternionic density matrices $\rho = \rho_\alpha + \epsilon \mathrm{j}\rho_\beta$ and the quaternionic Hamiltonian term $H(t) = H_\alpha + \mathrm{j}H_\beta$ where $L_i[\rho_\alpha] = H_\beta\epsilon\rho_\beta - \epsilon\rho_\beta^* H_\beta^*$.

6. The (n = 2)-dimensional case

Let us illustrate the above results with two-dimensional systems. In two-dimensional complex Hilbert spaces ($n = 2$) the most general complex density matrix is given by

$$\rho_\alpha = \begin{pmatrix} c & x + \mathrm{i}y \\ x - \mathrm{i}y & 1 - c \end{pmatrix}$$

with ρ_α implies that

$$1 \geqslant \sqrt{(2c-1)^2 + 4(x^2+y^2)}$$

to ensure positivity.

If $S_\nu : \mathcal{S}_2 \mapsto \mathcal{S}_2$ denotes the positive map $\rho_\alpha \mapsto \sigma_\nu \rho_\alpha \sigma_\nu$ with $\nu = 0, 1, 2, 3$, the decomposable map [18]

$$\rho_\alpha \mapsto \mathrm{T}_2[\rho_\alpha] := \frac{1}{2} \sum_{\nu=0}^{3} \varepsilon_\nu S_\nu[\rho_\alpha], \tag{34}$$

where $\varepsilon_\nu = 1$ for $\nu \neq 2$ and $\varepsilon_2 = -1$, which corresponds to the transposition. If we have $H_\alpha = 0$, $F_i = \sigma_\nu/\sqrt{2}$, $\nu = 1, 2, 3$ and

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

as Kossakowski matrix, the generator of this positive (not completely positive) map is given by

$$\frac{\partial}{\partial t}\gamma_t[\rho_\alpha] := L[\rho_\alpha] = \frac{1}{2}\left(\sum_{\nu=1}^{3} \varepsilon_\nu S_\nu[\rho_\alpha(t)] - \rho_\alpha(t)\right), \tag{35}$$

which reduces to

$$\gamma_t = \frac{1+\mathrm{e}^{-2t}}{2} I_2 + \frac{1-\mathrm{e}^{-2t}}{2}\mathrm{T}_2. \tag{36}$$

By applying this operator γ_t on ρ_α we obtain

$$\rho_\alpha(t) = \gamma_t \rho_\alpha(0) = \begin{pmatrix} c & x + \mathrm{i}\,\mathrm{e}^{-2t} y \\ x - \mathrm{i}\,\mathrm{e}^{-2t} y & 1-c \end{pmatrix},$$

and substituting $\rho_\alpha(t)$ into equation (35) one gets

$$L[\rho_\alpha] = 2\mathrm{i}y\,\mathrm{e}^{-2t} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the next subsections, we will describe this dynamics in terms of the complex projection of quaternionic unitary dynamics in a $(n = 2)$-dimensional quaternionic Hilbert space where the dimensionality of the operators coincides with double dimensionality of the quaternion numbers (i.e., they are represented by 2×2 quaternionic matrices). In particular, in subsection 6.1 the description will be given in terms of quaternionic mixed states and in subsection 6.2 the initial and final states will be quaternionic pure states. We recall that quaternionic states ρ are physically indistinguishable from their complex projections ρ_α, as long as we limit ourselves to consider complex observables only.

6.1. The case of quaternionic non-pure states

Let us consider the density matrices

$$\rho(0) = \begin{pmatrix} c & x + \mathrm{i}y \\ x - \mathrm{i}y & 1-c \end{pmatrix}$$

and

$$\rho(t) = \begin{pmatrix} c & x + \mathrm{i}\,\mathrm{e}^{-2t} y + \mathrm{j}p \\ x - \mathrm{i}\,\mathrm{e}^{-2t} y - \mathrm{j}p & 1-c \end{pmatrix},$$

as initial and final states of the quaternionic evolution map in a ($n = 2$)-dimensional quaternionic Hilbert space, where p has to be determined by imposing that $\rho(0)$ and $\rho(t)$ are unitarily similar (see equation (14)). Hence $\rho(0)$ and $\rho(t)$ must have the same eigenvalues. This implies that

$$p = y\sqrt{1 - \mathrm{e}^{-4t}}.$$

A quaternionic unitary evolution operator is given by

$$U(t) = (\cos\vartheta(t) + k\sin\vartheta(t))\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{37}$$

where, $\cos\vartheta(t) = \sqrt{\frac{1-\mathrm{e}^{-2t}}{2}}$ and $\sin\vartheta(t) = \sqrt{\frac{1+\mathrm{e}^{-2t}}{2}}$.

The quaternionic anti-Hermitian Hamiltonian operator can be obtained from equation (37) (see equation (19))

$$H(t) = -\left(\frac{\partial}{\partial t}U(t)\right)U^{\dagger}(t) = \mathrm{j}\frac{\mathrm{i}\,\mathrm{e}^{-2t}}{\sqrt{1-\mathrm{e}^{-4t}}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{j}H_{\beta}.$$

Moreover, in this case

$$\rho_{\beta} = p\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

thus

$$H_{\beta}^{*}\rho_{\beta} - \rho_{\beta}^{*}H_{\beta} = 2\mathrm{i}y\,\mathrm{e}^{-2t}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = L[\rho_{\alpha}].$$

6.2. The case of quaternionic pure states

Any ($n = 2$)-dimensional mixed state ρ_{α} can be purified adding to it a suitable purely quaternionic term $\mathrm{j}\rho_{\beta}$,

$$\rho = \rho_{\alpha} + \mathrm{j}\rho_{\beta} = \begin{pmatrix} c & x+\mathrm{i}y \\ x-\mathrm{i}y & 1-c \end{pmatrix} + \mathrm{j}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta}w \\ -\mathrm{e}^{\mathrm{i}\theta}w & 0 \end{pmatrix}. \tag{38}$$

In fact, by choosing

$$w = \sqrt{c(1-c) + x^2 + y^2}$$

in equation (38), the eigenvalues of ρ become 1 and 0 [13].

In particular putting $\theta = 0$ in equation (38), the initial and final states of our system become

$$\rho(0) = \begin{pmatrix} c & x+\mathrm{i}y+\mathrm{j}w \\ x-\mathrm{i}y-\mathrm{j}w & 1-c \end{pmatrix}$$

and

$$\rho(t) = \begin{pmatrix} c & x+\mathrm{i}\,\mathrm{e}^{-2t}y+\mathrm{j}r \\ x-\mathrm{i}\,\mathrm{e}^{-2t}y-\mathrm{j}r & 1-c, \end{pmatrix}$$

respectively, where r has to be determined by imposing that $\rho(0)$ and $\rho(t)$ are unitarily similar,which implies [13]

$$r = \sqrt{c(1-c) + x^2 + y^2(2 - \mathrm{e}^{-4t})}.$$

A quaternionic unitary evolution operator connecting $\rho(0)$ and $\rho(t)$ is

$$U(t) = (\cos\phi(t) + k\sin\phi(t))\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{39}$$

with $\cos\phi(t) = \left(\frac{r+w}{2r}\right)^{\frac{1}{2}}$ and $\sin\phi(t) = \left(\frac{r-w}{2r}\right)^{\frac{1}{2}}$.

The corresponding quaternionic anti-Hermitian Hamiltonian operator is (see equation (19))

$$H(t) = -\left(\frac{\partial}{\partial t}U(t)\right)U^{\dagger}(t) = j\,\frac{\mathrm{i}y\,\mathrm{e}^{-2t}}{r}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathrm{j}H_{\beta}.$$

Moreover, in this case

$$\rho_{\beta} = r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

and

$$H_{\beta}^{*}\rho_{\beta} - \rho_{\beta}^{*}H_{\beta} = 2\mathrm{i}y\,\mathrm{e}^{-2t}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = L[\rho_{\alpha}].$$

7. Conclusions

We have shown that QQM provides a useful tool to classify positive maps in CQM in agreement with the suggestions of [7]. Moreover, in many cases, positive maps in CQM are the complex projection of unitary maps in QQM. Hence, for many physical quantum systems the complex quantum mechanical description can be identified with complex projection of an underlying quaternionic quantum theory.

In this way the description of positive maps which are not completely positive in CQM can be understood in terms of QQM. However, we also have shown that for mixed states with non-maximal rank, which includes pure states, the QQM picture does not work.

Another open interesting problem of QQM is the following. In quaternionic vector spaces, the usual definition of Kronecker product of matrices does not hold, and the concept of tensor product of Hilbert spaces fails, because of the non-commutativity of the skew-field $\mathbb{Q}$. In order to overcome this difficulty, a concept of tensor product of quaternionic Hilbert modules has been proposed [19, 20], which allows one to describe composite systems on a mathematically well-founded basis; unfortunately, the results obtained in this way do not agree in the complex projection with those of standard quantum mechanics [21]. A different approach to composite systems in QQM is introduced in [22] but works only for very particular physical cases. The problem has to be solved because it constitutes the major obstruction for a formulation of quantum field theory in terms of quaternionic quantum theories.

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